The space L^2 on semi-infinite Grassmannian over finite field

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We construct a GL-invariant measure on a semi-infinite Grassmannian over a finite field, describe the natural group of symmetries of this measure, and decompose the space L^2 over the Grassmannian on irreducible representations. The spectrum is discrete, spherical functions on the Grassmannian are given in terms of the Al Salam–Carlitz orthogonal polynomials. We also construct an invariant measure on the corresponding space of flags.

1 Results of the paper

1.1. Group $\mathbf{GL}(\ell\oplus\ell^\circ)$. Let $\mathbb{F}_q$ be a finite field with q elements. Denote by ℓ the direct sum of countable number of $\mathbb{F}_q$. Denote by ℓ° the direct product of countable number of $\mathbb{F}_q$. Consider the linear space $\ell\oplus\ell^\circ$ equipped with the natural topology. Denote by $\mathbf{GL}(\ell\oplus\ell^\circ)$ the group of all continuous invertible linear operators $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $\ell\oplus\ell^\circ$. A description of this group is given below in Section 2.

There is a natural homomorphism $\theta:\mathbf{GL}(\ell\oplus\ell^\circ)\to\mathbb{Z}$ defined by

$$\theta\begin{pmatrix} a & b\\ c & d\end{pmatrix}:=\dim(\ell/a\ell)-\dim(\ker a)=-\bigl(\dim(\ell^\circ/d\ell^\circ)-\dim(\ker d)\bigr).$$

Denote by $\mathbf{GL}^0(\ell\oplus\ell^\circ)$ the kernel of this homomorphism.

1.2. Semi-infinite Grassmannian. For any infinite matrix T over $\mathbb{F}_q$ consider the subspace in $\ell\oplus\ell^\circ$ consisting of vectors $v\oplus vT\in\ell\oplus\ell^\circ$. Denote by $\mathcal{M}$ the set of all subspaces in $\ell\oplus\ell^\circ$ having such form. We say that a subspace M in $\ell\oplus\ell^\circ$ is *semi-infinite* if there exists $L\in\mathcal{M}$ such that

$$\alpha(L):=\dim L/(L\cap M)<\infty,\qquad \beta(L):=\dim M/(L\cap M)<\infty.$$

We say that

$$\mathrm{Dim}(L):=\alpha(L)-\beta(L).$$

is the *relative dimension* of L. Denote by $\mathbf{Gr}$ the set of all semi-infinite subspaces, by $\mathbf{Gr}^\alpha$ the set of subspaces of relative dimension α. For any $g\in\mathbf{GL}(\ell\oplus\ell^\circ)$ and semi-infinite L we have

$$\mathrm{Dim}(Lg)=\mathrm{Dim}\,L+\theta(g).$$

Therefore, $\mathbf{GL}^0(\ell\oplus\ell^\circ)$ acts on each $\mathbf{Gr}^\alpha$. This action is transitive.

1.3. The invariant measure on Grassmannian.

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Theorem 1.1 a) *There exists a unique up to a scalar factor finite* $\mathbf{GL}^0(\ell\oplus\ell^\circ)$*-invariant Borel measure* μ *on* $\mathbf{Gr}^0$*.*

b) *The restriction of the measure* μ *to the cell* $\mathcal{M}\simeq\mathbb{F}_q^{\infty\times\infty}$ *is the product measure on the countable product of* $\mathbb{F}_q$*.*

c) *If we normalize* μ *by the condition* $\mu(\mathcal{M})=1$*, then the total measure of the Grassmannian is*

$$\mu(\mathbf{Gr}^0)=\prod_{j=1}^{\infty}(1-q^{-j})^{-1}.$$

1.4. An intertwining operator. For $L\in\mathbf{Gr}^0$ consider the set Σ_L of all semi-infinite subspaces $K\in\mathbf{Gr}^0$ such that

$$\dim L/(L\cap K)=1 \qquad \dim K/(L\cap K)=1.$$

Proposition 1.2 *There is a unique probabilistic measure* ν_L *on* Σ_L *invariant with respect to the stabilizer of* L *in* $\mathbf{GL}(\ell\oplus\ell^\circ)$*.*

Theorem 1.3 a) *The operator*

$$\Delta f(L)=\int_{\Sigma_L} f(K)d\nu_L(K)$$

is a bounded self-adjoint $\mathbf{GL}(\ell\oplus\ell^\circ)$*-intertwining operator in* $L^2(\mathbf{Gr}^0,\mu)$*.*

b) *The spectrum of* Δ *is the set* $\{1,q^{-1},q^{-2},q^{-3},\dots\}$*.*

c) *The representations of* $\mathbf{GL}(\ell\oplus\ell^\circ)$ *in the eigenspaces*

$$\Delta f(L)=q^{-j}f(L) \tag{1.1}$$

are irreducible and pairwise nonequivalent.

1.5. Spherical functions on the Grassmannian. Denote by $\mathbf{P}\subset\mathbf{GL}(\ell\oplus\ell^\circ)$ the stabilizer of the subspace $0\oplus\ell^\circ$.

Proposition 1.4 *Orbits of* $\mathbf{P}$ *on* $\mathbf{Gr}^0$ *are enumerated by*

$$k:=\dim\bigl(L\cap(0\oplus\ell^\circ)\bigr)$$

($k=0,\,1,\,2,\dots$)*. Measures of orbits are*

$$\frac{q^{-k^2}}{\prod_{j=1}^{k}(1-q^{-j})^2}.$$

Theorem 1.5 a) *Any irreducible* $\mathbf{GL}(\ell\oplus\ell^\circ)$*-subrepresentation in* $L^2(\mathbf{Gr}^0)$ *contains a unique* $\mathbf{P}$*-invariant function.*

b) *The* $\mathbf{P}$*-invariant function in the eigenspace* $\Delta f=q^{-\alpha}f$ *as a function of* k *is given by the Al-Salam-Carlitz polynomial*

$$V_\alpha^{(1)}(q^k,q^{-1})=(-1)^\alpha q^{\alpha(\alpha-1)/2}{}_2\varphi_0\left[\begin{matrix}q^\alpha,q^k\\ \text{---}\end{matrix};q^{-1};q^{-\alpha}\right].$$

1.6. Measures on flags. Let $[\alpha,\beta]$ be a segment of $\mathbb{Z}$. We denote by $\mathbf{Fl}[\alpha,\beta]$ the space of all flags in $\ell\oplus\ell^\circ$ of the form

$$L_\alpha \subset L_{\alpha+1} \subset \cdots \subset L_\beta, \qquad L_j \in \mathbf{Gr}^j.$$

We also allow infinite segments $(-\infty,\infty)$, $[\alpha,\infty)$, $[-\infty,\beta]$.

Proposition 1.6 *For any segment $I\subset\mathbb{Z}$ there is a unique $\mathbf{GL}(\ell\oplus\ell^\circ)$-invariant measure on $\mathbf{Fl}[I]$.*

1.7. On groups of infinite-dimensional matrices over finite fields. Now there is a relatively well-developed and relatively well-understood representation theory of infinite-dimensional classical groups and infinite symmetric groups. In a strange way, this is not so for groups of infinite matrices over finite fields. In 1976 Skudlarek [14] tried to extend Thoma's work on characters of infinite symmetric group to $\mathrm{GL}(\infty,\mathbb{F}_q)$, but his result was negative (the latter group has no non-trivial characters). Some positive results were obtained by Vershik and Kerov [17], [18]. Their main object is the group of infinite matrices having only finite number of nonzero elements under the diagonal. Other group of works is [3]– [5], some our standpoints arise from [5].

In a certain sense the present paper is the analog of works on

1) projective limits of complex Grassmannians [13], classical groups, and symmetric spaces, [10] (also [11], Section 2.10), [1];

2) projective limits of symmetric groups, [6];

3) projective limits of p-adic Grassmannians, [12].

However, our construction is not precisely a projective limit. Also, pre-limit constructions in 1)-2) are analysis on groups and symmetric spaces. In our case, the pre-limit objects are representations of $\mathrm{GL}(N,\mathbb{F}_q)$ induced from maximal parabolics (the real analog is [16]), the explicit decomposition of such representations was obtained by Zhornovsky and Zelevinsky [19] and by Stanton, [15].

1.8. Notation.
— $\mathbb{F}_p^\times$ is the multiplicative group of the field $\mathbb{F}_q$.
— $\#S$ is the number of elements of a finite set S.

2 Linear operators in the spaces ℓ, ℓ°, $\ell\oplus\ell^\circ$

Here we examine the group $\mathbf{GL}(\ell\oplus\ell^\circ)$ of invertible linear operators in $\ell\oplus\ell^\circ$. In particular, we get toy analogs of Fredholm index theory and of Banach's inverse operator theorem.

2.1. The spaces ℓ and ℓ°. Denote by ℓ the direct sum of countable number of $\mathbb{F}_q$. In other words, ℓ is the linear space of formal linear combinations

$$v = v_1e_1 + v_2e_2 + \dots,$$

where e_j are basis vectors and $v_j \in \mathbb{F}_q$ are 0 for all but finite number of v_j. We equip ℓ with the discrete topology.

Denote by ℓ° the direct product of countable number of $\mathbb{F}_q$. In other words, ℓ is the linear space of all formal linear combinations

$$w = w_1 f_1 + w_2 f_2 + \dots,$$

where f_j are basis elements and sequences $w_j \in \mathbb{F}_q$ are arbitrary. We equip $\mathbb{F}_q$ with the discrete topology. Then the direct product ℓ° becomes a compact totally disconnected topological space. We have obvious embedding $\ell \to \ell^\circ$, evidently the image is dense.

The spaces ℓ and ℓ° are dual in the following sense:

— any linear functional on ℓ has the form $h_w(v) = \sum v_j w_j$ for some $w \in \ell^\circ$;

— any continuous linear functional on ℓ° has the form $h_v(w) = \sum w_j v_j$ for some $w \in \ell^\circ$.

Lemma 2.1 *Any closed subspace in ℓ° is an annihilator of some subspace in ℓ.*

Proof. Here can refer to Pontryagin duality, see, e.g., [8] Theorem 27.

2.2. Linear operators in ℓ and ℓ°. Any square infinite matrix A over $\mathbb{F}_q$ determines a linear operator $\ell \to \ell^\circ$ given by $v \mapsto vA$,

$$\begin{pmatrix} v_1 & v_2 & \dots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \\ = \begin{pmatrix} v_1 a_{11} + v_2 a_{21} + \dots & v_1 a_{21} + v_2 a_{22} + \dots & \dots \end{pmatrix}$$

Lemma 2.2 *A matrix A determines a linear operator $\ell \to \ell$ if and only if each row of A contains only finite number of non-zero elements.*

This is obvious (rows of A are images of the basis vectors e_j). By duality we get the following statement.

Lemma 2.3 *A matrix A determines a continuous linear operator $\ell^\circ \to \ell^\circ$ if and only if each column of A contains only finite number of non-zero elements.*

The transposition $A \mapsto A^t$ is an antiisomorphism ($(AB)^t = B^t A^t$) of the ring of linear operators in ℓ and the ring of continuous linear operators in ℓ°.

Lemma 2.4 *A matrix A determines a linear operator $\ell^\circ \to \ell$ if and only if A contains only finite number of non-zero elements.*

Proof. Let A be an operator $\ell^\circ \to \ell$. By Lemma 2.3 each column of A contains only finite number of non nonzero elements. The transposed operator also acts $\ell^\circ \to \ell$, therefore each row of A contains only finite number of non-zero elements. Assume that A contains an infinite number of nonzero elements.

Consider a sequence $f_{k_1}, f_{k_2}, \dots$ such that $Af_{k_j} \neq 0$. Set of nonzero coordinates of each Af_{k_j} is finite, therefore we can choose a subsequence $Af_{k_{j_m}}$ such that the sets of nonzero coordinates are mutually disjoint. Then $A\sum_m f_{k_{j_m}} \notin \ell$. $\square$

2.3. Groups of invertible operators. Denote by $\mathbf{GL}(\ell)$ and $\mathbf{GL}(\ell^\circ)$ groups of invertible continuous linear operators in ℓ and ℓ° respectively.

Proposition 2.5 *A linear operator $A : \ell \to \ell$ (see Lemma 2.2) is invertible if and only if it satisfies the following additional conditions:*

— any finite collection of rows of the matrix A is linear independent;

— columns of A are linear independent (i.e., all finite and countable linear combinations of columns are nonzero).

We emphasize that by Lemma 2.2 *any countable linear combinations of columns are well-defined.*

Proof. The first condition is equivalent to $\ker A = 0$. The second condition means that a linear functional on ℓ vanishing on $\operatorname{im} A$ is zero, i.e. $\operatorname{im} A = \ell$. Under these two conditions the operator A is invertible. $\square$

By the duality we get the following corollary

Proposition 2.6 *A linear operator $A : \ell^\circ \to \ell^\circ$ (see Lemma 2.3) is invertible if only if it satisfies the following additional conditions:*

— any finite collection of columns of the matrix A is linear independent

— rows of A are linear independent (again, we admit countable linear combinations of rows).

2.4. Fredholm operators in ℓ. We say that an operator $A : \ell \to \ell$ is a *Fredholm operator* if

$$\dim \ker A < \infty \qquad \operatorname{codim} \operatorname{im} A < \infty.$$

Its *index* is

$$\operatorname{ind} A := \operatorname{codim} \operatorname{im} A - \dim \ker A.$$

Lemma 2.7 *An operator $A : \ell \to \ell$ is Fredholm if and only if it can be represented in the form*

$$A = g_1 J g_2,$$

where $g_1, g_2 \in \mathbf{GL}(\ell)$, and J is a $(\alpha + \infty) \times (\beta + \infty)$ block matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with finite α and β.

Moreover,

$$\beta = \dim \ker A \qquad \alpha = \operatorname{codim} \operatorname{im} A$$

PROOF. 'If' is obvious.

Let A be Fredholm. We take an arbitrary operator $h_2 \in \mathbf{GL}(\ell)$ sending $\ker A$ to a subspace of the form $\oplus_{j<\beta}\mathbb{F}_q e_j$. Next, take $h_1 \in \mathbf{GL}(\ell^\circ)$ sending the annihilator of $\operatorname{im} A$ to $\oplus_{j<\alpha}\mathbb{F}_q f_j$. Then $h_1^t A h_2^{-1}$ has the form $\begin{pmatrix}0&0\\0&B\end{pmatrix}$ with invertible B. □

Proposition 2.8 *Let $A : \ell \to \ell$ be Fredholm. Let $K : \ell \to \ell$ be a continuous operator of finite rank. Then $A + K$ is a Fredholm operator of the same index.*

PROOF. Without loss of generality we can assume that A has canonical form $\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Since $\operatorname{rk} K$ is finite, it has only finite number of nonzero columns. Therefore, without loss of generality, we can assume that K has only one nonzero column. After this a verification is straightforward. □

Proposition 2.9 *Let A be an operator $\ell \to \ell$. Assume that there are operators B, $C : \ell \to \ell$ such that*

$$AB = 1 + K, \qquad CA = 1 + L,$$

where K and L have finite ranks. Then A is Fredholm.

PROOF. Indeed,

$$\ker A \subset \ker(1 + K), \qquad \operatorname{im} A \supset \operatorname{im}(1 + L).$$

Proposition 2.10 *An operator $A : \ell \to \ell$ is Fredholm if and only if there are finite α, β and an invertible matrix* ('dilatation')

$$\begin{pmatrix}p&q\\r&A\end{pmatrix}$$

of size $(\alpha + \infty) \times (\beta + \infty)$. Moreover,

$$\operatorname{ind}(A) = \alpha - \beta.$$

PROOF. Arrow $\Rightarrow$ easily follows from Lemma 2.7. Let us prove $\Leftarrow$.

Represent $\begin{pmatrix}p&q\\r&A\end{pmatrix}^{-1}$ as a block $(\beta+\infty)\times(\alpha+\infty)$ matrix $\begin{pmatrix}x&y\\z&B\end{pmatrix}$. Then

$$AB = 1 - ry, \qquad BA = 1 - zq,$$

and we apply Proposition 2.9. □

Proposition 2.11 *Let A, B be Fredholm operators in ℓ. Then AB is Fredholm and*

$$\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

PROOF. We include A and B to invertible matrices according Proposition 2.10,

$$\widetilde{A} = \begin{pmatrix} p & q \\ r & A \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} x & y \\ z & B \end{pmatrix}.$$

Let their sizes be $(\alpha + \infty) \times (\beta + \infty)$ and $(\gamma + \infty) \times (\delta + \infty)$. If $\alpha = \delta$, then we write

$$\widetilde{A}\widetilde{B} = \begin{pmatrix} p & q \\ r & A \end{pmatrix} \begin{pmatrix} x & y \\ z & B \end{pmatrix} = \begin{pmatrix} * & * \\ * & AB + ry \end{pmatrix}.$$

The product has size $(\gamma + \infty) \times (\beta + \infty)$. Therefore $AB + rz$ is Fredholm of index $\gamma - \beta$. Therefore, AB is Fredholm of the same index.

If $\alpha > \delta$, we change $\widetilde{B}$ to

$$\widetilde{B}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & z & B \end{pmatrix},$$

where the unit block has size $\alpha - \delta$ and multiply $\widetilde{A}\widetilde{B}'$. If $\alpha < \delta$, then we enlarge $\widetilde{A}$. □

2.5. Fredholm operators in ℓ°. We say that an operator $A : \ell^\circ \to \ell^\circ$ is *Fredholm* if $\ker A$ is finite-dimensional, $\operatorname{im} A$ has finite codimension[2]. The *index* is $(\operatorname{codim} \operatorname{im} A - \dim \ker A)$ as above.

Proposition 2.12 *An operator $A : \ell^\circ \to \ell^\circ$ is Fredholm if and only if A^t is Fredholm in ℓ,*

$$\operatorname{ind} A^t = -\operatorname{ind} A$$

2.6. Linear operators in $\ell \oplus \ell^\circ$. Now consider the space $\ell \oplus \ell^\circ$, denote the standard basis in ℓ by e_j, and the standard basis in ℓ° by f_j.

The dual space is $\ell^\circ \oplus \ell$. Note that $\ell \oplus \ell^\circ$ is a locally compact Abelian group, therefore we have a possibility to refer to the Pontryagin duality. In particular,

Lemma 2.13 *Any closed subspace in $\ell \oplus \ell^\circ$ is the annihilator of a closed subspace in $\ell^\circ \oplus \ell$.*

We write continuous linear operators in $\ell \oplus \ell^\circ$ as block $(\infty + \infty) \times (\infty + \infty)$ matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(and regard elements of $\ell \oplus \ell^\circ$ as block row matrices). We also use the notation

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

By Lemmas 2.2–2.4 such matrices satisfy the conditions:

[2]Since the space ℓ° is compact, $\operatorname{im} A$ is closed for any continuous A

a) Any row of a contains finite number of nonzero elements;

b) Any column of d contains finite number of nonzero elements;

c) c has only finite number of nonzero elements.

2.7. The group $\mathbf{GL}(\ell\oplus\ell^\circ)$. We denote by $\mathbf{GL}(\ell\oplus\ell^\circ)$ the group of all continuous invertible linear operators in $\ell\oplus\ell^\circ$.

Lemma 2.14 *If* $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathbf{GL}(\ell\oplus\ell^\circ)$*, then a and d are Fredholm.*

PROOF. Let $\begin{pmatrix} p & q\\ r & t\end{pmatrix}$ be the inverse matrix. Then

$$\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} p & q\\ r & t\end{pmatrix}=\begin{pmatrix} ap+br & *\\ * & cq+dt\end{pmatrix};$$
$$\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} p & q\\ r & t\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} pa+qc & *\\ * & rb+td\end{pmatrix}.$$

Matrices d, r are finite, therefore br, cq, qc, rb have finite rank. By Proposition 2.9, a and d are Fredholm □

2.8. Generators of $\mathbf{GL}(\ell\oplus\ell^\circ)$. Next, we define certain subgroups in $\mathbf{GL}(\ell\oplus\ell^\circ)$.

1) The group $\mathrm{GL}(2\infty,\mathbb{F}_q)$ is the group of finite invertible matrices, i.e., matrices g such that $g-1$ has only finite number of nonzero elements. We represent $\mathrm{GL}(2\infty,\mathbb{F}_q)$ as a union of subgroups $\mathrm{GL}(2n,\mathbb{F}_q)$ consisting of block $(n+\infty)\times(n+\infty)$ of the form

$$\begin{pmatrix} u & 0 & v & 0\\ 0 & 1 & 0 & 0\\ w & 0 & x & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The group $\mathrm{GL}(2\infty,\mathbb{F}_q)$ is the inductive limit

$$\mathrm{GL}(2\infty,\mathbb{F}_q)=\lim_{n\to\infty}\mathrm{GL}(2n,\mathbb{F}_q).$$

2) The 'parabolic' subgroup $\mathbf{P}$ consists of matrices $\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$ such that $a\in\mathbf{GL}(\ell)$, $d\in\mathbf{GL}(\ell^\circ)$.

3) Next consider an operator J defined by

$$\begin{aligned} &e_iJ=e_{i-1}, \text{where } i\geqslant 2;\\ &e_1J=f_1;\\ &f_kJ=f_{k+1}.\end{aligned}\tag{2.1}$$

Denote by $\mathcal{Z}\subset\mathbf{GL}(\ell\oplus\ell^\circ)$ the cyclic group generated by J.

Now we present two descriptions of the group $\mathbf{GL}(\ell\oplus\ell^\circ)$.

Theorem 2.15 *The group* $\mathbf{GL}(\ell \oplus \ell^\circ)$ *is generated by the subgroups* $\mathrm{GL}(2\infty, \mathbb{F}_q)$, $\mathbf{P}$*, and* $\mathcal{Z}$*.*

PROOF. Let $g \in \mathbf{GL}(\ell \oplus \ell^\circ)$. We choose k such that for $g' := J^k g$ the block g'_{11} has index 0. A multiplication of the form

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} := g'', \qquad \text{where } u,\, v \in \mathbf{GL}(\ell), \tag{2.2}$$

allows to get

$$g''_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(the size of the matrix is $(k+\infty) \times (k+\infty)$). We write g''_{21} as a block $\infty \times (k+\infty)$-matrix $g''_{21} = \begin{pmatrix} p & q \end{pmatrix}$. We set

$$g''' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & 1 \end{pmatrix} g'',$$

the size of the first factor is $(k + \infty + \infty) \times (k + \infty + \infty)$, it is contained in $\mathrm{GL}(2\infty, \mathbb{F}_q)$. In this way, we get

$$g'''_{21} = \begin{pmatrix} p & 0 \end{pmatrix}$$

(other blocks are the same). The rank of p is k. We choose k linear independent rows of p and permute them with first k rows of g'''_{11} (this corresponds to a multiplication by a certain element of $\mathrm{GL}(2\infty, \mathbb{F}_q)$). Thus we get a new matrix g^{IV} with

$$g^{IV}_{11} = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$$

Repeating the operation (2.2), we can get $g^{V}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g^{V}_{21} = \begin{pmatrix} p' & 0 \end{pmatrix}$. After this we kill p' by

$$g^{VI} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p' & 0 & 1 \end{pmatrix} g^{V}$$

and get a matrix of the form

$$g^{VI} = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}.$$

This matrix is contained in $\mathbf{P}$. $\square$

The proof gives slightly more strong statement, which is necessary for us below.

Corollary 2.16 *Any element of* $\mathbf{GL}^0$ *can be represented as*

$$g = hqr, \qquad \textit{where } h \in \mathbf{GL}(\ell),\ p \in \mathrm{GL}(2\infty, \mathbb{F}_q),\ r \in \mathbf{P}.$$

2.9. The inverse operator theorem.

Theorem 2.17 *If a continuous linear operator g in $\ell \oplus \ell^\circ$ is bijective, then g is invertible.*

Proof. Since the block c is finite, the image of $0 \oplus \ell^\circ$ is contained in some subspace

$$H = \Big(\bigoplus_{j \leqslant k} \mathbb{F}_q e_j\Big) \oplus \ell^\circ.$$

The subspace $\ell^\circ g$ is compact, therefore it is a closed in H. For any closed subspace $L \subset H$ the dimension H/L is finite or continual. On the other hand the dimension $(\ell \oplus \ell^\circ)/\ell^\circ$ is countable, therefore the dimension $(\ell \oplus \ell^\circ)/\ell^\circ g$ is countable. Thus, $\dim(H/\ell^\circ g)$ is finite. Applying an appropriate element of $\mathrm{GL}(2\infty, \mathbb{F}_q)$ we can put $\ell^\circ g$ to position

$$\Big(\bigoplus_{j \leqslant m} \mathbb{F}_q e_j\Big) \oplus \ell^\circ \qquad \text{or} \qquad \bigoplus_{i \geqslant \alpha} \mathbb{F}_q f_i.$$

Applying an appropriate power J^s we can assume $\ell^\circ g = \ell^\circ$. The map $g : \ell^\circ \to \ell^\circ g$ is continuous. Therefore the inverse map $g^{-1} : \ell^\circ g \to \ell^\circ$ is continuous. Since $(\ell \oplus \ell^\circ)/\ell^\circ g$ is discrete, the map g^{-1} is automatically continuous on the whole $\ell \oplus \ell^\circ$. □

Proposition 2.18 *Let g be a continuous operator in $\ell \oplus \ell^\circ$, $\ker g = 0$, and $\operatorname{im} g$ be dense. Assume that*

$$\dim \ell^\circ/(\ell^\circ \cap \ell^\circ g)$$

is finite. Then g is invertible.

Remark. The last condition is necessary. Consider the spaces $V = \ell \oplus \ell \oplus \ell^\circ$ and $W = \ell \oplus \ell^\circ \oplus \ell^\circ$ (both spaces can be identified with $\ell \oplus \ell^\circ$). Consider the operator $A : V \to W$ defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $\ker A = 0$ and $\operatorname{im} A := \ell \oplus \ell^\circ \oplus \ell$ is dense in W. But A is not invertible. □

Proof. As in the proof of Theorem 2.17 we can assume $\ell^\circ g = \ell^\circ$. Thus $(\ell \oplus \ell^\circ)/\ell^\circ g$ is a space with discrete topology. Since the image of g is dense, g must be surjective. □

2.10. The subgroup $\mathbf{GL}^0(\ell \oplus \ell^\circ)$.

Theorem 2.19 *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ range in $\mathbf{GL}(\ell \oplus \ell^\circ)$.*

a) $\operatorname{ind} a + \operatorname{ind} d = 0$.

b) *The map*

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{ind} a$$

is a homomorphism $\mathbf{GL}(\ell \oplus \ell^\circ) \to \mathbb{Z}$.

c) *The homomorphism* θ *is also determined by the conditions*

$$\theta(g) = 0, \qquad \text{if } g \in \mathbf{P}; \tag{2.3}$$

$$\theta(h) = 0, \qquad \text{if } h \in \mathrm{GL}(2\infty, \mathbb{F}_q); \tag{2.4}$$

$$\theta(J) = 1. \tag{2.5}$$

PROOF. b) Let $\begin{pmatrix} p & q \\ r & t \end{pmatrix}$ be the inverse matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} ap + br & * \\ * & * \end{pmatrix}.$$

By Proposition 2.11, ap is Fredholm and $\operatorname{ind} ap = \operatorname{ind} a + \operatorname{ind} p$. The matrix r is finite, br has finite rank. Therefore $\operatorname{ind}(ap + br) = \operatorname{ind} ap$.

c) is obvious.

a) $\widetilde{\theta}(g) := -\operatorname{ind} d$ also is a homomorphism $\mathbf{GL}(\ell \oplus \ell^\circ) \to \mathbb{Z}$. We have $\widetilde{\theta}(g) = \theta(g)$ on generators (2.3)–(2.5). Therefore $\widetilde{\theta} = \theta$. □

We denote by $\mathbf{GL}^\circ(\ell \oplus \ell^\circ)$ the kernel of the homomorphism θ.

3 The semi-infinite Grassmannian

3.1. The Grassmannian. We use alternative notation for the following subspaces in $\ell \oplus \ell^\circ$:

$$V := \ell \oplus 0, \qquad W := 0 \oplus \ell^\circ.$$

We define the *semi-infinite Grassmannian* **Gr** as the set of all subspaces in $\ell \oplus \ell^\circ$ having the form

$$L = Vg, \qquad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(\ell \oplus \ell^\circ), \tag{3.1}$$

i.e., **Gr** is the $\mathbf{GL}(\ell \oplus \ell^\circ)$-orbit of ℓ°.

We define the *relative dimension* of L as

$$\operatorname{Dim}(L) := \operatorname{ind}(a) = \theta(g).$$

By definition,

$$\operatorname{Dim} Lh = \operatorname{Dim} L + \theta(h), \qquad L \in \mathbf{Gr}, h \in \mathbf{GL}(\ell \oplus \ell^\circ).$$

We denote by $\mathbf{Gr}^0$ the $\mathbf{GL}^0(\ell \oplus \ell^\circ)$-orbit of V. It consists of elements of **Gr** having relative dimension 0.

The following lemma implies that $\operatorname{Dim} L$ does not depend on the choice of g in (3.1).

Lemma 3.1 *Let $L \in \mathbf{Gr}$. Denote by $p(L)$ the projection of L to V along W. Then*

$$\operatorname{Dim} L = \dim(L \cap W) - \dim(V/p(L)).$$

Proof. The subspace (3.1) consists of vectors

$$va \oplus vb, \qquad \text{where } v \text{ ranges in } V. \tag{3.2}$$

We have $L \cap W = (\ker a)b$. The operator b does not vanish on $\ker a$, otherwise $\ker g \neq 0$. Therefore $\dim L \cap W = \dim \ker a$.

On the other hand $p(L) = \operatorname{im} a$. □

Consider a linear operator $T : \ell \to \ell^\circ$. Consider its graph $\operatorname{graph}(T) \subset \ell \oplus \ell^\circ$, i.e., the set of vectors of the form $v \oplus vT$. Denote by $\mathcal{M}$ the set of all such subspaces in $\ell \oplus \ell^\circ$.

Lemma 3.2 a) $\mathcal{M} \subset \mathbf{Gr}_0$.

b) *If a in* (3.1) *is invertible, then $T = a^{-1}b$.*

c) *If $a + Tc$ is invertible, then*

$$\operatorname{graph}(T)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{graph}\bigl((a+Tc)^{-1}(b+Td)\bigr) \tag{3.3}$$

d) *Let $a + Tc$ be invertible. Consider the images $\widetilde{V}$, $\widetilde{W}$ of subspaces V and W under $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, i.e. spaces consisting of vectors $va \oplus vc$ and $wb \oplus wc$, where v ranges in V and w ranges in W. Then the subspace* $\operatorname{graph} T \subset V \oplus W$ *is a graph of an operator $\widetilde{V} \to \widetilde{W}$, whose matrix is $(a + Tc)^{-1}(b + Td)$.*

Proof. a) We set $g = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$.

b) In (3.2) we set $v' = av$ and get $v' \oplus v'a^{-1}b$.

c) We write

$$\begin{pmatrix} v & vT \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} v(a+Tc) & v(b+Td) \end{pmatrix}$$

and set $v' = v(a + Tc)$.

d) is a reformulation of c).

□

3.2. A characterization of elements of Gr.

Lemma 3.3 a) *Let $L = Vg \in \mathbf{Gr}$, M be a subspace in L,* $\dim L/M < \infty$. *Then $M \in \mathbf{Gr}$.*

b) *Let $L \in \mathbf{Gr}$, let $N \supset L$ be a subspace in $\ell \oplus \ell^\circ$ such that* $\dim N/L < \infty$. *Then $N \in \mathbf{Gr}$.*

c) *If $L \supset M$ are elements of $\mathbf{Gr}$, then*

$$\operatorname{Dim} L = \operatorname{Dim} M + \dim L/M.$$

PROOF. Let us prove a). Mg^{-1} is a subspace in V. Consider an invertible operator h in $V \simeq \ell$ such that

$$Mg^{-1}h^{-1} = \bigoplus_{j\geqslant k} \mathbb{F}_q e_j$$

Then $M = VJ^k hg$. □

Proposition 3.4 *For any subspace $L \in \mathbf{Gr}$ there exists $M \in \mathcal{M}$ such that* $\dim L/(L\cap M) < \infty$, $\dim M/(L \cap M) < \infty$.

PROOF. Let $L = V\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. First, consider an (infinite-dimensional) complement $X \subset V$ to $\ker a$ and consider the subspace $L' := Xg$. We have $L' \cap W = 0$.

Consider the projection $p(L')$ to V along W. Consider a (finite-dimensional) complement Z of $p(L')$ in V. Set $L'' := L' \oplus Z$. We have $p(L'') = V$ and $L'' \cap W = -0$. Thus $L'' \in \mathcal{M}$. □

3.3. Atlas on Gr. Consider finite subsets Ω and Ξ in $\mathbb{N}$. Consider the following subspaces $V[\Omega, \Xi]$ and $W[\Omega,\Xi]$ in $\ell \oplus \ell^\circ$:

$$V[\Omega,\Xi] := \Bigl(\bigoplus_{i\notin\Omega} \mathbb{F}_q e_i\Bigr) \oplus \Bigl(\bigoplus_{j\in\Xi} \mathbb{F}_q f_j\Bigr), \tag{3.4}$$

$$W[\Omega,\Xi] := \Bigl(\bigoplus_{i\in\Omega} \mathbb{F}_q e_i\Bigr) \oplus \Bigl(\bigoplus_{j\notin\Xi} \mathbb{F}_q f_j\Bigr) \tag{3.5}$$

(sums are topological direct sums). We have

$$\ell \oplus \ell^\circ = V[\Omega,\Xi] \oplus W[\Omega,\Xi].$$

Denote by $\mathcal{M}[\Omega,\Xi]$ the set of all subspaces, which are graphs of operators $V[\Omega,\Xi] \to W[\Omega,\Xi]$. Notice that

$$V = V[\varnothing,\varnothing], \qquad W = W[\varnothing,\varnothing], \qquad \mathcal{M} = \mathcal{M}[\varnothing,\varnothing].$$

Theorem 3.5 $\mathbf{Gr} = \bigcup_{\Omega,\Xi\subset\mathbb{Z}} \mathcal{M}[\Omega,\Xi]$.

Consider a union $\mathbb{N} \sqcup \mathbb{N}'$ of two copies of $\mathbb{N}$, we assume that the first copy enumerates vectors e_j and the second copy vectors f_i. Denote by $S(2\infty)$ the group of finite permutations $\mathbb{N} \sqcup \mathbb{N}'$. For each element of $S(2\infty)$ we write 0-1 matrix (an element of $\mathrm{GL}(2\infty, \mathbb{F}_q)$) in the usual way. We also add the following permutation to the group $S(2\infty)$:

$$\dots \to e_2 \to e_1 \to f_1 \to f_2 \to \dots$$

It corresponds to the element $J \in \mathbf{GL}(\ell \oplus \ell^\circ)$. Thus we get the extended group $\mathbb{Z} \ltimes S(2\infty)$.

Lemma 3.6 *An element $\sigma\in\mathbb{Z}\ltimes S(2\infty)\subset\mathbf{GL}(\ell\oplus\ell^\circ)$ send a chart $\mathcal{M}[\Omega,\Xi]$ to the chart $\mathcal{M}[\Omega_1,\Xi_1]$, where*

$$\bigl[(\mathbb{N}\setminus\Omega)\cup\Xi\bigr]\sigma=(\mathbb{N}\setminus\Omega_1)\cup\Xi_1.$$

This is obvious.

Proof of Theorem 3.5. The chart $\mathcal{M}[\varnothing,\varnothing]$ is contained in $\mathbf{Gr}$. By Lemma 3.6 all charts $\mathcal{M}[\Omega,\Xi]$ are contained in $\mathbf{Gr}$.

Let us prove $\subset$. Let $L\in\mathcal{M}[\Omega,\Xi]$. Let M have codimension 1 in L. We wish to show that M is contained in some $\mathcal{M}[\Omega',\Xi']$. Consider the projection $p(M)$ of M to $V(\Omega,\Xi)$ along $W(\Omega,\Xi)$. It is determined by an equation

$$\sum_{j\notin\Omega}\alpha_j v_j+\sum_{i\in\Xi}\beta_i w_i=0,\quad\text{where}\ \sum_{j\notin\Omega}v_je_j+\sum_{i\in\Xi}w_if_j\in L$$

If some $\alpha_j\ne0$, we can take $\Omega'=\Omega\cup\{j\}$, $\Xi'=\Xi$. If some $\beta_i\ne0$, we can take $\Omega'=\Omega$, $\Xi'=\Xi\setminus\{i\}$.

Next, let $L\in\mathcal{M}[\Omega,\Xi]$ has codimension 1 in K. We wish to show that K is contained in some set $\mathcal{M}[\Omega',\Xi']$. We add a vector

$$h=\sum\beta_je_j+\sum\gamma_if_i$$

to L. But this vector in not canonical, we can eliminate part of coordinates and take

$$h=\sum_{j\in\Omega}\gamma_je_j+\sum_{i\notin\Xi}\delta_if_i.$$

If some $\gamma_j\ne0$, we can exclude $\{j\}$ from Ω. On the other hand, if some $\beta_i\ne0$, we can add i to Ξ.

It remains to refer to Proposition 3.4. □

4 The measure on Grassmannian

4.1. The measure μ. Note that each chart $\mathcal{M}[\Omega,\Xi]$ is a direct product of countable family of $\mathbb{F}_q$. We equip each chart by the product measure.

Theorem 4.1 a) *There is a unique σ-finite measure μ on $\mathbf{Gr}$ coinciding with the standard measure on each chart $\mathcal{M}[\Omega,\Xi]$.*

b) *This measure μ is $\mathbf{GL}(\ell\oplus\ell^\circ)$-invariant.*

c) *The measure of $\mathbf{Gr}^0$ is*

$$\mu(\mathbf{Gr}^0)=\prod_{j=1}^{\infty}(1-q^{-j})^{-1}. \tag{4.1}$$

d) *Denote by $\mathcal{O}_k$ the set of all $L\in\mathbf{Gr}^0$ such that* $\dim L\cap W=k$. *Then*

$$\mu(\mathcal{O}_k)=\frac{q^{-k^2}}{\prod_{j=1}^k(1-q^{-j})^2}. \tag{4.2}$$

The proof occupies the rest of this section.

4.2. The measure on ℓ°. We equip $\mathbb{F}_q$ with uniform probability measure (the measure of each point is $1/q$). We equip

$$\ell^\circ = \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q \times \dots$$

with the product measure. Note that this measure is the Haar measure on the compact Abelian group ℓ°. We have natural maps $\pi_n : \ell^\circ \to \mathbb{F}_q^n$ given by

$$(v_1, v_2, v_3 \dots) \to (v_1, \dots, v_n).$$

A *cylindric subset* in ℓ° is a set which can be obtained as a preimage of a subset S in $\mathbb{F}_q^n$ for some n. Finite unions and finite intersections of cylindric subsets are cylindric.

Proposition 4.2 *The group $\mathbf{GL}(\ell^\circ)$ acts on ℓ° preserving the measure. Images and preimages of cylindric sets are cylindric.*

This is obvious.

4.3. The measure on projective space. Consider the projective space $\mathbb{P}\ell^\circ = (\ell^\circ \setminus 0)/\mathbb{F}_q^\times$. Consider the map $\ell^\circ \to \mathbb{P}\ell^\circ$ defined a.s. Denote by $\varkappa$ the image of the measure under this map. We have a quotient of an invariant *finite* measure by an invariant partition ($w \sim w'$ iff $w' = sw$, where $s \in \mathbb{F}_q^\times$). Therefore we get the following trivial statement.

Proposition 4.3 *The measure $\varkappa$ on $\mathbb{P}\ell^\circ$ is $\mathbf{GL}(\ell^\circ)$-invariant.*

4.4. The measure on $\mathcal{M}$. The set $\mathcal{M}$ also is an infinite product of $\mathbb{F}_q$, we equip it with the product measure. Let us examine transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : T \mapsto (a + Tc)^{-1}(b + Td)$$

of $\mathcal{M}$, see Lemma 3.2.

Lemma 4.4 *The group $\mathbf{P}$ acts on $\mathcal{M}$ by measure preserving transformations. They send cylindric sets to cylindric sets.*

Proof. Since $c = 0$, we have transformations

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : T \mapsto a^{-1}b + a^{-1}Td$$

defined on the whole $\mathcal{M}$. In fact we have the product of transformations of 3 types corresponding to matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

The transformations $T \mapsto a^{-1}T$ act column-wise. Precisely, denote by $t^{(1)}$, $t^{(2)}$, ..., the columns of T. Each column ranges in the space ℓ°, thus we can write $\mathcal{M} = \ell^\circ \times \ell^\circ \times \dots$. Our transformation is

$$(t^{(1)}, t^{(2)}, \dots) \to (a^{-1}t^{(1)}, a^{-1}t^{(2)}, \dots)$$

and we refer to Proposition 4.2.

Similarly, the transformations $T \mapsto Td$ act row-wise.

The transformations $T \mapsto T + b$ are simply translations on the compact Abelian group $\mathcal{M}$. $\square$.

Lemma 4.5 *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}^0(\ell \oplus \ell^\circ)$*. Then the set* R *of all* $T \in \mathcal{M}$ *such that* $\det(a + Tc) = 0$ *is cylindric. The map* $T \mapsto (a + Tc)^{-1}(b + zd)$ *is measure preserving outside* R*.*

PROOF. First, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2\infty, \mathbb{F}_q)$. We represent each block of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a block $(N + \infty) \times (N + \infty)$-matrix. For sufficiently large N this matrix has the following structure

$$\begin{pmatrix} a_{11} & 0 & b_{11} & 0 \\ 0 & 1 & 0 & 0 \\ c_{11} & 0 & d_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then our transformation has the form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \mapsto$$

$$\left[\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}\right]^{-1}\left[\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} d_{11} & 0 \\ 0 & 1 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} a_{11} + T_{11}c_{11} & 0 \\ T_{21}c_{11} & 1 \end{pmatrix}^{-1}\begin{pmatrix} b_{11} + T_{11}d_{11} & T_{12} \\ T_{21}d_{11} & T_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} (a_{11} + T_{11}c_{11})^{-1} & 0 \\ -T_{21}c_{11}(a_{11} + T_{11}c_{11})^{-1} & 1 \end{pmatrix}\begin{pmatrix} b_{11} + T_{11}d_{11} & T_{12} \\ T_{21}d_{11} & T_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} (a_{11} + T_{11}c_{11})^{-1}(b_{11} + T_{11}d_{11}) & (a_{11} + T_{11}c_{11})^{-1}T_{12} \\ T_{21}\left[-c_{11}(a_{11} + T_{11}c_{11})^{-1}(b_{11} + T_{11}d_{11}) + d_{11}\right] & -T_{21}c_{11}(a_{11} + T_{11}c_{11})^{-1}T_{12} + T_{22} \end{pmatrix}$$

We observe that the set R is determined by the equation $\det(a + T_{11}d_{11}) = 0$. Evidently, this set is cylindric. Fix T_{11} outside 'surface' $\det(a+T_{11}d_{11}) = 0$. The set $T_{11} = const$ is a product of 3 spaces $\mathbb{F}_q^{N\times\infty}$, $\mathbb{F}_q^{\infty\times N}$, $\mathbb{F}_q^{\infty\times\infty}$ corresponding blocks T_{12}, T_{21}, T_{22} of T.

The transformations $T_{12} \mapsto (a_{11} + T_{11}c_{11})^{-1}T_{12}$ are column-wise (columns have length N).

The transformations $T_{21} \mapsto T_{21}\left[-c_{11}(a_{11}+T_{11}c_{11})^{-1}(b_{11}+T_{11}d_{11})+d_{11}\right]$ are row-wise. Rows have length N. The matrix in square brackets is invertible. Indeed,

$$0 \neq \det\begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix} = \det\begin{pmatrix} a_{11}+T_{11}c_{11} & b_{11} \\ c_{11}+T_{11}d_{11} & d_{11} \end{pmatrix} =$$
$$= \det(a+zc)\det\left[-c_{11}(a_{11}+T_{11}c_{11})^{-1}(b_{11}+T_{11}d_{11})+d_{11}\right]$$

It remains to examine transformations

$$T_{22} \to -T_{21}c_{11}(a_{11}+T_{11}c_{11})^{-1}T_{12}+T_{22}.$$

First, let $k, l > N$. The new matrix element t_{kl} of T depends only on elements $t_{\alpha,\beta}$ with $\alpha \leqslant k$, $\beta \leqslant l$. Therefore the preimage of any cylindric set is cylindric. We can apply the same argument to the inverse transformation and get that the image of any cylindric set is cylindric.

Next, for fixed T_{12}, T_{21} the transformation of T_{22} is a translation. Therefore it preserves the measure.

Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}^0(\ell \oplus \ell^\circ)$ be arbitrary. We refer to Corollary of Theorem 2.15, recall that each element of $\mathbf{GL}^0$ can be represented as

$$g = hqr, \qquad \text{where } h \in \mathbf{GL}(\ell),\ p \in \mathrm{GL}(2\infty, \mathbb{F}_q),\ r \in \mathbf{P}.$$

By Lemma 4.4 h and r act on $\mathcal{M}$ by measure-preserving transformations defined everywhere, hence the lemma reduces to the case examined above. $\square$

Each chart $\mathcal{M}[\Omega, \Xi]$ is the space of square matrices, we equip a chart $\mathcal{M}[\Omega, \Xi]$ by the product measure $\mu_{\Omega,\Xi}$.

Lemma 4.6 *The measures $\mu_{\Omega,\Xi}$ and $\mu_{\Omega',\Xi'}$ coincide on $\mathcal{M}[\Omega, \Xi] \cap \mathcal{M}[\Omega', \Xi']$.*

Proof. Consider charts $\mathcal{M}[\varnothing, \varnothing]$, $\mathcal{M}[\Omega, \Xi]$. Consider a transformation of $\ell \oplus \ell^\circ$ sending $V[\varnothing, \varnothing]$ to $V[\Omega, \Xi]$ and $W[\varnothing, \varnothing]$ to $W[\Omega, \Xi]$ (we can take a finite permutation of coordinates, see Lemma 3.6)). Lemma 3.2.d gives an expression for coordinate of graph(T) in the chart $\mathcal{M}[\Omega, \Xi]$. It is linear-fractional. By Lemma 4.5 this map is measure-preserving. $\square$

Corollary 4.7 *The measure μ is well-defined.*

Let $g \in \mathbf{GL}(\ell \oplus \ell^\circ)$. To each $T \in \mathcal{M}[\varnothing, \varnothing]$ we assign the coordinate of the subspace graph(T) in $\mathcal{M}[\Omega, \Xi]$. Thus we get a partially defined map $\mathcal{M}[\varnothing, \varnothing] \to \mathcal{M}[\Omega, \Xi]$

Lemma 4.8 *This map is measure preserving.*

Proof. Indeed, according Lemma 3.2, this map is linear-fractional. $\square$

Corollary 4.9 *The measure μ is $\mathbf{GL}(\ell \oplus \ell^\circ)$-invariant.*

4.5. Uniqueness of μ. We have a natural Borel structure on each chart and therefore a natural Borel structure on $\mathbf{Gr}$.

Proposition 4.10 a) *A* $\mathbf{GL}^0(\ell\oplus\ell^\circ)$*-invariant finite or* σ*-finite measure on* $\mathbf{Gr}^j$ *coincides with* μ *up to a constant factor.*

b) *A* $\mathbf{GL}(\ell\oplus\ell^\circ)$*-invariant* σ*-finite measure* ν *on* $\mathbf{Gr}$ *defined on all Borel sets coincides with* μ *up to a constant factor.*

PROOF. The restriction of ν to $\mathcal{M}[\Omega,\Xi]\simeq\mathbb{F}_q^{\infty\times\infty}$ must be invariant with respect to all translations. Therefore it is the product measure up to a constant factor. By Lemma 4.6 these factors are common for all Ω, Ξ with $\#\Omega=\#\Xi$. Therefore the measures on each $\mathbf{Gr}^j$ coincides with μ up to a constant factor.

The transformation J identifies $\mathbf{Gr}^j$ and $\mathbf{Gr}^{j+1}$. $\square$

4.6. Finite Grassmannians. Denote by Gr_m^k the set of all k-dimensional subspaces in the linear space $\mathbb{F}_q^m$. Let Gr_m be set of all subspaces in $\mathbb{F}_q^m$,

$$\mathrm{Gr}_m=\bigsqcup_{j=0}^m \mathrm{Gr}_m^k.$$

We need some simple facts (see, e.g. [2]) about such Grassmannians.

1) Number of elements in $\mathrm{GL}(m,\mathbb{F}_q)$ is

$$\gamma_m:=\prod_{j=1}^m(q^m-q^j)=q^{m^2}\prod_{j=1}^m(1-q^{-j}).$$

2) Denote by $P=P_n\subset\mathrm{GL}(2n,\mathbb{F}_q)$ the stabilizer of the subspace $0\oplus\mathbb{F}_q^n$ in $\mathbb{F}_q^{2n}$. The number of elements of Gr_{2n}^n is

$$\frac{\#\mathrm{GL}(2n,\mathbb{F}_q)}{\#P}=\frac{\gamma_{2n}}{\gamma_n^2\,q^{n^2}}=q^{n^2}\cdot\frac{\prod_{j=1}^n(1-q^{-n-j})}{\prod_{j=1}^n(1-q^{-j})}. \tag{4.3}$$

Indeed, Gr_{2n}^n is a $\mathrm{GL}(2n,\mathbb{F}_q)$-homogeneous space, the subgroup P consists of $(n+n)\times(n+n)$-matrices $\begin{pmatrix}a&b\\0&d\end{pmatrix}$, where a, $d\in\mathrm{GL}(n,\mathbb{F}_q)$ and b is arbitrary.

3) Consider the set $\mathcal{O}_k(n)$ of all subspaces $L\in\mathrm{Gr}_{2n}^n$ that have k-dimensional intersection with $W:=0\oplus\mathbb{F}_q^n$. The number of elements of this set is

$$\#\mathcal{O}_k(n)=\frac{\gamma_n^2\,q^{n^2}}{\gamma_n^2\gamma_{n-k}^2\,q^{4k(n-k)}}=q^{n^2}\cdot\frac{q^{-k^2}\prod_{j=n-k+1}^n(1-q^{-j})^2}{\prod_{j=1}^k(1-q^{-j})^2} \tag{4.4}$$

Indeed, the stabilizer Q of a pair (L,M) of subspaces stabilizes also $L\cap M$, $L+M$ and the decomposition of $(L+M)/(L\cap M)$ into a direct sum

$$(L+M)/(L\cap M)=L/(L\cap M)\oplus M/(L\cap M)$$

Therefore Q consists of invertible matrices of size $k+(n-k)+(n-k)+k$ having the following structure

$$\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

The desired number is $\frac{\#P}{\#Q}$. $\square$

4) We regard $\mathbb{F}_q^{2n}$ as a direct sum $V_n \oplus W_n := \mathbb{F}_q^n \oplus \mathbb{F}_q^n$, set

$$V := \bigoplus_{j=1}^{N} \mathbb{F}_q e_j, \qquad W := \bigoplus_{j=1}^{N} \mathbb{F}_q f_j.$$

Next, for subsets Ω, $\Xi \in \{1, 2, \dots, n\}$ such that $\#\Omega = \#\Xi$, we define subspaces $V_n[\Omega, \Xi]$, $W_n[\Omega, \Xi]$ as above, (3.4)–(3.5). As above, we define a chart $\mathcal{M}_n[\Omega, \Xi] \subset \mathrm{Gr}_{2n}^n$ as the set of graphs of operators $V_n[\Omega, \Xi] \to W_n[\Omega, \Xi]$.

We define a uniform measure μ_n on Gr_{2n}^n by the assumption: for any set $S \subset \mathrm{Gr}_{2n}^n$,

$$\mu_n(S) = \frac{\#S}{q^{n^2}}.$$

In particular, the measure of any chart $\mathcal{M}_n[\Omega, \Xi]$ is 1.

4.7. The map $\mathbf{Gr} \to \mathrm{Gr}_n$. Consider the the following subspaces X_n and Y_n in $\ell \oplus \ell^\circ$:

$$X_n := \bigoplus_{j>n} \mathbb{F}_q f_j \qquad Y_n := \bigoplus_{i \leqslant n} \mathbb{F}_q e_i \oplus \bigoplus_{j \in \mathbb{N}} \mathbb{F}_q f_j.$$

We have an obvious isomorphism

$$Y_n / X_n \simeq \mathbb{F}_q^{2n}.$$

For $L \in \mathbf{Gr}^0$ we define $\pi_n(L) \in \mathrm{Gr}_{2n}$ as the image of $L \cap Y_n$ under the map $Y_n \to Y_n/X_n$.

Remark. A dimension of $\pi_n(L)$ can be arbitrary between 0 and $2n$. $\square$

Lemma 4.11 a) *If Ω, $\Xi \subset \{1, 2, \dots, n\}$, and $L \in \mathcal{M}[\Omega, \Xi]$, then $\dim \pi_n(L) = n$.*

b) *Moreover, under the same condition*

$$\pi_n(\mathcal{M}[\Omega, \Xi]) = \mathcal{M}_n[\Omega, \Xi]$$

and the image of the measure μ on $\mathcal{M}[\Omega, \Xi]$ is the measure μ_n on $\mathcal{M}[\Omega, \Xi]$.

c) *Any cylindric subset in $\mathcal{M}[\Omega, \Xi]$ is a π_n-preimage of a subset in $\mathcal{M}_n[\Omega, \Xi]$ for sufficiently large n.*

This is straightforward.

Denote by $\mathcal{U}_n$ the following subset in $\mathbf{Gr}^0$:

$$\mathcal{U}_n = \bigcup_{\substack{\Omega, \Xi \subset \{1,2,\dots,n\} \\ \#\Omega = \#\Xi}} \mathcal{M}[\Omega, \Xi]$$

Let φ be a function on Gr_{2n}^n. Consider the function φ^* on $\mathbf{Gr}^0$ given by

$$\varphi^*(L) := \begin{cases} \varphi(\pi_n(L)), & \text{if } L \in \mathcal{U}_n; \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

Remark. The preimage of Gr_{2n}^n under the map π_n is larger than $\mathcal{U}_n$. $\square$

Denote by $\mathcal{F}_n$ the set of functions on $\mathbf{Gr}^0$ that can be obtained in this way. Obviously, $\mathcal{F}_{n+1} \supset \mathcal{F}_n$.

Lemma 4.12 a) *The map* $\pi_n : \mathcal{U}_n \to \mathrm{Gr}_{2n}^n$ *commutes with action of* $\mathrm{GL}(2n, \mathbb{F}_q)$.

b) $\bigcup_n \mathcal{U}_n = \mathbf{Gr}^0$.

c) $\bigcup_n \mathcal{F}_n$ *is dense in* $L^2(\mathbf{Gr}^0)$.

This is obvious.

Corollary 4.13 *The total measure of* $\mathbf{Gr}^0$ *is finite and equals* (4.1).

Proof.

$$\mu(\mathbf{Gr}^0) = \lim_{n\to\infty} \mathcal{U}_n = \lim_{n\to\infty} \mu_n(\mathrm{Gr}_{2n}^n),$$

and we apply (4.3). $\square$

Corollary 4.14 *The measure of* $\mathcal{O}_k$ *is given by* (4.2).

Proof. Let $L \in \mathbf{Gr}^0$. We have $W \subset Y_n$. On the other hand for large n, we have $L \cap X_n = 0$ (since $L \cap W$ is finite-dimensional). Therefore, starting sufficiently large n we have

$$\dim(\pi_n(L) \cap W_n) = \dim L \cap W,$$

Therefore

$$\mu(\mathcal{O}_k) = \lim_{n\to\infty} \mathcal{O}_k(n)$$

and we apply (4.4).

5 Decomposition of $L^2(\mathbf{Gr}^0)$ and Al-Salam–Carlitz polynomials

The group $\mathbf{GL}^0(\ell\oplus\ell^\circ)$ acts in $L^2(\mathbf{Gr}^0)$ by changes of variables

$$\rho(g)F(L)=F(Lg).$$

Here we decompose this representation.

5.1. Operator of average. Let $L\in\mathbf{Gr}$. Denote by

— $\Sigma_L^{\downarrow}$ the set of all subspaces $K\subset L$ such that $\dim(L/K)=1$;

— $\Sigma_L^{\uparrow}$ the set of all spaces $N\in\mathbf{Gr}$ such that $N\supset L$ and $\dim N/L=1$;

— $\Sigma_L^{\updownarrow}$ the set of subspaces $M\in\mathbf{Gr}$ such that

$$\dim L/(L\cap M)=1,\qquad \dim M/(L\cap M)=1.$$

Lemma 5.1 *Let $L\in\mathbf{Gr}$, $\mathcal{P}_L\subset\mathbf{GL}(\ell\oplus\ell^\circ)$ be the stabilizer L in $\mathbf{GL}(\ell\oplus\ell^\circ)$.*

a) *There is a unique $\mathcal{P}(L)$-invariant probability measure $\sigma_L^{\updownarrow}$ on the space $\Sigma_L^{\updownarrow}$.*

b) *There is a unique $\mathcal{P}(L)$-invariant probability measure $\sigma_L^{\downarrow}$ on the space $\Sigma_L^{\downarrow}$.*

c) *There is a unique $\mathcal{P}(L)$-invariant probability measure $\sigma_L^{\uparrow}$ on the space $\Sigma_L^{\uparrow}$.*

Proof. b) Without loss of generality we set $L=V$. The group $\mathcal{P}$ consists of matrices $\begin{pmatrix}a&0\\c&d\end{pmatrix}$. Subspaces of codimension 1 in V are determined by equations $\sum\alpha_j v_j=0$, where $\sum v_je_j\in V$ and $\alpha=(\alpha_1,\alpha_2,\dots)$ is in ℓ°. Therefore the set of subspaces K is the projective space $\mathbb{PF}_q^\infty\simeq(\ell^\circ\setminus 0)/\mathbb{F}_q^\times$ and equipped with a canonical measure $\varkappa$, see Proposition 4.3. Matrices $\begin{pmatrix}1&0\\c&d\end{pmatrix}$ fix all elements of V and therefore act trivially on the projective space $\mathbb{P}\ell^\circ$. The group of matrices $\begin{pmatrix}a&0\\0&1\end{pmatrix}$ acts by projective transformations and therefore preserve the measure $\varkappa$.

a) Again, without loss of generality we set $L=V$. The quotient space $(\ell\oplus\ell^\circ)/V$ is isomorphic to ℓ°. Therefore overspaces $M\supset L$ are enumerated by points of $\mathbb{P}\ell^\circ$.

c) The existence of the measure follows from b) and c). The space $\Sigma_L^{\updownarrow}$ is $\mathcal{P}$-homogeneous, therefore the invariant measure is unique. $\square$

We define the operator Δ in $L^2(\mathbf{Gr}^0)$ by

$$\Delta f(L)=\int_{M\in\Sigma_L^{\updownarrow}} f(M)\,d\nu_L^{\updownarrow}(M).$$

Proposition 5.2 a) *The operator Δ is a self-adjoint bounded operator in $L^2(\mathbf{Gr}^0)$.*

b) *The operator Δ is $\mathbf{GL}^0$-intertwining.*

PROOF. The statement b) follows from the $\mathcal{P}_L$-invariance of the measure ν_L.

Consider the measure $\nu^\updownarrow$ on $\mathbf{Gr}^0 \times \mathbf{Gr}^0$ defined by

$$\nu^\updownarrow(S) = \int_{\mathbf{Gr}^0} \nu_L^\updownarrow(S \cap (\{L\} \times \mathbf{Gr}^0)\, d\mu(L), \qquad S \subset \mathbf{Gr}^0 \times \mathbf{Gr}^0,$$

here we regard the measure $\{\nu_L^\updownarrow\}$ as a measure on $\mathbf{Gr}^0$. We identify any fiber $\{L\} \times \mathbf{Gr}^0 \subset \mathbf{Gr}^0 \times \mathbf{Gr}^0$ with $\mathbf{Gr}^0$. This measure satisfies

$$\langle f, \Delta g \rangle_{L^2(\mathbf{Gr}^0)} = \int_{\mathbf{Gr}^0 \times \mathbf{Gr}^0} f(L)\, \overline{g}(M)\, d\nu^\updownarrow(L, M) \tag{5.1}$$

We need the following lemma.

Lemma 5.3 *The measure $\nu^\updownarrow$ is symmetric with respect to transposition of factors in $\mathbf{Gr}^0 \times \mathbf{Gr}^0$.*

PROOF OF LEMMA. The measure $\nu^\updownarrow$ is invariant with respect to the diagonal action of $\mathbf{GL}^0(\ell \oplus \ell^\circ)$ on $\mathbf{Gr}^0 \times \mathbf{Gr}^0$. Therefore its projection to the second factor is $\mathbf{Gr}^0$-invariant, by Proposition 4.10 it coincides with μ. Consider conditional measures on the fibers $\mathbf{Gr}^0 \times \{K\}$. They must be $\mathcal{P}_L$-invariant a.s., hence they coincide with $\nu_L^\updownarrow$. This implies the desired statement. □

END OF PROOF OF PROPOSITION 5.2. Let us look to (5.1) as to an abstract expression. Let X be a space with a probability measure μ, let S be a measure on the product $X \times X$ and projections of S to both factors coincide with μ. Then the formula (5.1) determines a so-called *Markov operator.* Such operators automatically satisfy $\|\Delta\| \leqslant 1$, see, e.g., [9], Theorem VIII.4.2. Since the measure S is symmetric with respect to the transposition of factors, Δ is self-adjoint. □

5.2. P-invariant functions. Recall that $\mathbf{P}$ is the stabilizer of the subspace $W = 0 \oplus \ell^\circ$ in $\mathbf{GL}^0(\ell \oplus \ell^\circ)$. Orbits $\mathcal{O}_k$ of $\mathbf{P}$ on the Grassmannian $\mathbf{Gr}^0$ are the following sets:

$$L: \quad \dim(L \cap W) = k.$$

Denote by $\mathcal{H}$ the space of $\mathbf{P}$-fixed functions on $\mathbf{Gr}^0$, it consists of functions constant on orbits $\mathcal{O}_k$.

Proposition 5.4 *The $\mathbf{GL}^0(\ell \oplus \ell^\circ)$-cyclic span of $\mathcal{H}$ is the whole $L^2(\mathbf{Gr}^0)$.*

5.3. Proof of Proposition 5.4. Consider the group $\mathrm{GL}(2n, \mathbb{F}_q)$ and the homogeneous space $\mathrm{Gr}_{2n}^n = \mathrm{GL}(2n, \mathbb{F}_q)/P$, where $P = P_n$ is the group of $(n+n)\times(n+n)$-matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. The pair $\mathrm{GL}(2n, \mathbb{F}_q) \supset P$ is spherical (see, e.g. [2], 8.6.5). The representation of $\mathrm{GL}(2n, \mathbb{F}_q)$ on the space of functions on Gr_{2n}^n was decomposed in [19], [15]. It is a sum of $n+1$ irreducible subrepresentations, each subrepresentation has a unique P-fixed vector (by the Frobenius reciprocity).

Such vectors can be regarded as functions of $k = \dim L \cap W$. P-fixed vectors are given by q-Hahn polynomials[3] Q_0, Q_1, ..., Q_n:

$$Q_j(q^{-k}; q^{-n-1}, q^{n-1}; n; q) := {}_3\varphi_2 \left(\begin{matrix} q^{-j}, q^{j-2n-1}, q^{-k} \\ q^{-n}, q^{-n} \end{matrix}; q; q \right).$$

q-Hahn polynomials are regarded as functions of the discrete variable $k = 0$, $1, \ldots, n$.

We use the standard notation for basic hypergeometric functions

$$\begin{aligned} {}_r\varphi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right) &:= \\ &:= \sum_{k=0}^{\infty} \frac{(a_1; q)_k \ldots (a_r; q)_k}{(b_1; q)_k (b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)k(k-1)/2} \frac{z^k}{(q; q)_k}, \end{aligned} \tag{5.2}$$

and

$$(a; q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1}).$$

Consider the difference operator

$$\mathcal{L}y(k) = B(k)y(k+1) - (B(k) + D(k))y(k) + D(k)y(k-1), \tag{5.3}$$

where

$$B(k) = (1 - q^{k-n})^2; \tag{5.4}$$

$$D(k) = (1 - q^{-2n-2})(1 - q^k)^2 \tag{5.5}$$

The q-Hahn polynomials are eigenfunctions of $\mathcal{L}$:

$$\mathcal{L}Q_j(k) = (1 - q^{-j})(1 - q^{j-2n-1})\, Q_j(k).$$

Consider the function $I_n(L)$ on Gr_{2n}^n, which equals 1 on $\mathcal{M}_n[\varnothing, \varnothing]$ and 0 otherwise

Lemma 5.5 *The* $\mathrm{GL}(2n, \mathbb{F}_q)$*-cyclic span of* I_n *is the whole space of functions on* Gr_{2n}^n.

PROOF. Assume that there is an irreducible subrepresentation Y orthogonal to I_n. It contains a P-fixed function $Q_j(k)$, which also must be orthogonal to I_n. This means that $Q_j(0) = 0$.

By (5.4), $B_j(k)$ is non-zero for all $k = 0, 1, \ldots, n-1$. By (5.3) we can express

$$y_j(k+1) = s_k \cdot y(k) + t_k \cdot y(k-1);$$
$$y(1) = s_0 y(0),$$

[3] On q-Hahn polynomials, see, e.g., [7].

where s_k, t_k are some coefficients. Therefore $Q_j(0) = 0$ implies $Q_j(k) = 0$ for all k. $\square$

PROOF OF PROPOSITION 5.4. Now consider the characteristic function I_∞ of the chart $\mathcal{M}[\varnothing, \varnothing] \subset \mathbf{Gr}_{2n}^n$. This function is contained in the subspace $\mathcal{F}_n$ (see Subsection 4.7) and has the form $I_n^*(L)$, see (4.5). By Lemma 5.5 the cyclic span of I_n is the whole space of functions on Grassmannian. By Lemma 4.12.a, the $\mathrm{GL}(2n, \mathbb{F}_q)$-cyclic span of I_∞ is $\mathcal{F}_n$. Therefore, $\mathrm{GL}(2\infty, \mathbb{F}_q)$-cyclic span of I_∞ is $\bigcup_n \mathcal{F}_n$. The latter subspace is dense in $L^2(\mathbf{Gr}^0)$, see Lemma 4.12.d. $\square$

5.4. Intersections of Σ_L with P-orbits. Let $L \in \mathcal{O}_k$. The set Σ_L have intersections with $\mathcal{O}_{k-1}$, $\mathcal{O}_k$, $\mathcal{O}_{k+1}$. Let us find $\nu_L^\updownarrow$-measures of these intersections.

Lemma 5.6

$$\nu_L^\updownarrow(\Sigma_L \cap \mathcal{O}_{k-1}) = (1 - q^{-k})^2; \tag{5.6}$$

$$\nu_L^\updownarrow(\Sigma_L \cap \mathcal{O}_k) = q^{-k} - q^{-2k} - q^{-2k-1}; \tag{5.7}$$

$$\nu_L^\updownarrow(\Sigma_L \cap \mathcal{O}_{k+1}) = q^{-2k-1}. \tag{5.8}$$

PROOF. We will choose a subspace $K \subset L$ of codimension 1. Next, we choose an overspace $M \supset K$ such that $\dim M/K = 1$.

STEP 1. Without loss of generality we can assume

$$L = \bigoplus_{j>k} \mathbb{F}_q e_j \oplus \bigoplus_{i \leqslant k} \mathbb{F}_q f_i$$

and

$$L \cap W = \bigoplus_{i \leqslant k} \mathbb{F}_q f_i. \tag{5.9}$$

A subspace M is determined by a linear equation

$$\sum_{j>k} \alpha_j v_j + \sum_{i \leqslant k} \beta_i w_i = 0, \quad \text{where} \sum_{j>k} v_j e_j + \sum_{i \leqslant k} w_i f_j \in V.$$

CASE 1. All $\beta_i = 0$. Then $M \cap W = L \cap W$. The probability of this event is q^{-k}.

CASE 2. There exists $\beta_i \neq 0$. Then $W \cap M$ is the subspace in (5.9) determined by the equation $\sum_{i \leqslant k} \beta_i w_i = 0$. Thus $\dim M \cap L = k-1$. The probability of this event is $1 - q^{-k}$.

STEP 2. Next, we choose $K \supset M$.

In Case-1 we by rotation set the subspace K to the position

$$K = \bigoplus_{j>k+1} \mathbb{F}_q e_j \oplus \bigoplus_{i \leqslant k} \mathbb{F}_q f_i.$$

We add a vector

$$h = \sum_{j \leqslant k+1} x_j e_j + \sum_{j>k} y_j f_j$$

to K. Again, there are two cases.

Case 1.1. All $x_j = 0$. Then $M \cap W = \mathbb{F}_q h \oplus \bigoplus_{i \leqslant k} \mathbb{F}_q f_i$. Therefore $M \in \mathcal{O}_{k+1}$. Conditional probability of this event is q^{-k-1}.

Case 1.2. There exists $x_j \neq 0$. Then $M \cap W = K \cap W$ and M falls to $\mathcal{O}_k$. The conditional probability of this event is $1 - q^{-k-1}$.

In Case-2 we put the subspace K to the position

$$K = \bigoplus_{j>k} \mathbb{F}_q e_j \oplus \bigoplus_{i \leqslant k-1} \mathbb{F}_q f_i,$$

add a vector

$$h = \sum_{j \leqslant k} x_j e_j + \sum_{j > k-1} y_j f_j$$

to K. Repeating the same consideration as in Cases 1.1-1.2 we get

Case 2.1. With conditional probability q^{-k} we fall to $\mathcal{O}_k$.

Case 2.2. With conditional probability $(1 - q^{-k})$ we fall to $\mathcal{O}_{k-1}$.

Compiling 4 sub-cases we get (5.6)–(5.8). $\square$

5.5. Difference operator. Now let us restrict the operator Δ to the subspace $\mathcal{H}$ of $\mathbf{P}$-invariant functions. The space $\mathcal{H}$ can be regarded as the space of complex sequences

$$x = (x(0), x(1), x(2), \dots)$$

with inner product

$$\langle x, y \rangle = \sum_{j=0}^{\infty} \frac{q^{-k^2}}{\prod_{j=1}^{k} (1 - q^{-j})^2} \cdot x(k) \overline{y}(k).$$

Proposition 5.7 *The restriction of Δ to $\mathcal{H}$ is the following difference operator*

$$\mathcal{L} y(k) = (1 - q^{-k})^2 y(k-1) + (2q^{-k} - q^{-2k} - q^{-2k-1}) y(k) + q^{-2k+1} y(k+1)$$

Proof. The statement follow from Lemma 5.6. $\square$

This difference Sturm–Liouville problem is known, the solutions are Al-Salam–Carlitz II polynomials

$$V_j^{(1)}(q^k, q^{-1}) = (-1)^j q^{j(j-1)/2} {}_2\varphi_0 \left(\begin{matrix} q^j, q^k \\ \text{---} \end{matrix}; q^{-1}; q^{-j} \right)$$

the corresponding eigenvalues are q^{-j}.

6 Measures on flag manifolds

Denote by $\mathbf{Gr}^\alpha$ the set of all subspaces of relative dimension α. The operator $J \in \mathbf{GL}(\ell \oplus \ell^\circ)$ send $\mathbf{Gr}^\alpha$ to $\mathbf{Gr}^{\alpha+1}$.

6.1. Measures on finite flags. Let $L \in \mathbf{Gr}$. Let the measures $\nu_L^\uparrow$ and $\nu_L^\downarrow$ be the same as above (Lemma 5.1). We regard them as measures on $\mathbf{Gr}$. Now we define the measure $\nu^\uparrow_{[\alpha,\alpha+1]}$ on $\mathbf{Gr}^\alpha \times \mathbf{Gr}^{\alpha+1}$ by

$$\nu^\uparrow_{[\alpha,\alpha+1]}(S) = \int_{\mathbf{Gr}^\alpha} \nu_L^\uparrow\bigl(S \cap (\{L\} \times \mathbf{Gr}^{\alpha+1})\bigr)\, d\mu(L)$$

The measure $\nu^\uparrow_{[\alpha,\alpha+1]}$ is supported by the pairs of subspaces

$$L \in \mathbf{Gr}^\alpha, \quad K \in \mathbf{Gr}^{\alpha+1}, \qquad L \subset K. \tag{6.1}$$

We also define the measure $\nu^\downarrow_{[\alpha,\alpha-1]}$ on $\mathbf{Gr}^\alpha \times \mathbf{Gr}^{\alpha-1}$ by

$$\nu^\downarrow_{[\alpha,\alpha-1]}(S) = \int_{\mathbf{Gr}^{\alpha 1}} \nu_L^\downarrow\bigl(S \cap (\{L\} \times \mathbf{Gr}^{\alpha})\bigr)\, d\mu(L)$$

Lemma 6.1 a) *The measure $\nu^\uparrow_{[\alpha,\alpha+1]}$ is a unique finite $\mathbf{GL}^0$-invariant measure supported by flags* (6.1)*.*

b) *Transposition of factors in $\mathbf{Gr}^\alpha \times \mathbf{Gr}^{\alpha-1}$ send $\nu^\downarrow_{[\alpha,\alpha-1]}$ to $\nu^\uparrow_{[\alpha-1,\alpha]}$.*

Proof is the same as proof of Lemma 5.3. □

For this reason we do not write arrows in superscripts of $\nu_{[\alpha,\alpha+1]}$.

Next, we define a measure $\nu_{[\alpha,\alpha+2]}$ on $\mathbf{Gr}^\alpha \times \mathbf{Gr}^{\alpha+1} \times \mathbf{Gr}^{\alpha+2}$ by

$$\nu_{[\alpha,\alpha+2]}(S) = \int_{\mathbf{Gr}^\alpha \times \mathbf{Gr}^{\alpha+1}} \nu_K^\uparrow\bigl(\{L\} \times \{K\} \times \mathbf{Gr}^{\alpha+2}\bigr)\, d\nu_{[\alpha,\alpha+2]}(L,K).$$

We get a measure supported by the space of flags

$$L \subset K \subset N, \qquad L \in \mathbf{Gr}^\alpha, K \in \mathbf{Gr}^{\alpha+1}, M \in \mathbf{Gr}^{\alpha+2}.$$

Iterating this operation we get measures $\nu_{[\alpha,\beta]}$ supported by the space $\mathbf{Fl}[\alpha,\beta]$ of flags

$$L_\alpha \subset L_{\alpha-1} \subset \dots \subset L_\beta, \qquad L_j \in \mathbf{Gr}^j$$

By construction, this measure is invariant with respect to the group $\mathbf{GL}^0(\ell \oplus \ell^\circ)$. Evidently it is the unique invariant probabilistic measure.

6.2. Measures on a space of infinite flags. If a segment $[\gamma,\delta]$ contains a segment $[\alpha,\beta]$. Then there is a forgetting map $\mathbf{Fl}[\gamma,\delta] \to \mathbf{Fl}[\alpha,\beta]$. Since the measure $\nu_{[\gamma,\delta]}$ is invariant, its pushforward is invariant and therefore coincides with $\nu_{[\alpha,\beta]}$. Therefore we get a $\mathbf{GL}(\ell \oplus \ell^\circ)$-invariant measure on the space $\mathbf{Fl}(-\infty,\infty)$ of complete flags

$$\dots L_{-1} \subset L_0 \subset L_1 \subset \dots, \qquad L_j \in \mathbf{Gr}^j.$$

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